LAWS OF PROBABILITY

Theorem 1. ***If A and B are events associated with a random experiment having sample space S and if $A \subset B$, then***

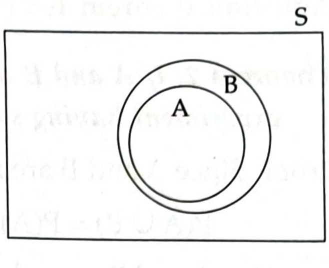
(i) $P(A) \le P(B)$ ***(ii)*** $P(B - A) = P(B) - P(A)$.

Proof. (*i*) Consider any arbitrary elementary event $w_i \in A$, then $w_i \in B$ $(\because A \subset B, \text{ given})$

$$\Rightarrow \Sigma P(w_i) \text{ for all } w_i \in A \le \Sigma P(w_j) \text{ for all } w_j \in B$$

$$\Rightarrow P(A) \le P(B).$$

(*ii*) From the adjoining Venn diagram, we have



$$B = A \cup (B - A) \quad \ldots(1)$$

Note that A and B – A are mutually exclusive, so by axiom (*iii*), we have

$$P(A \cup (B - A)) = P(A) + P(B - A)$$

$$\Rightarrow P(B) = P(A) + P(B - A) \quad \text{(using (1))}$$

$$\Rightarrow P(B - A) = P(B) - P(A).$$

For example, let S be the sample space associated with tossing a fair coin 3 times, then

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, which has 8 equally likely outcomes.

Let A = getting three heads = {HHH} and

B = getting atleast two heads = {HHH, HHT, HTH, THH}.

We note that $P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$ and $P(B) = \frac{n(B)}{n(S)} = \frac{4}{8}$.

Also we note that $A \subset B$ and B – A = {HHT, HTH, THH},

so $P(B - A) = \frac{n(B - A)}{n(S)} = \frac{3}{8}$.

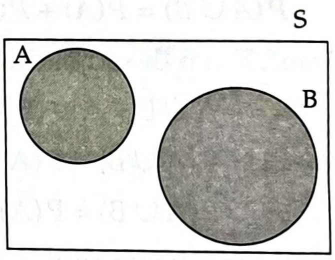
By the theorem proved above, we should have

$$P(B - A) = P(B) - P(A)$$

i.e. $\frac{3}{8} = \frac{4}{8} - \frac{1}{8}$, which is true.

Mutually exclusive events

If A and B are mutually exclusive events associated with a random experiment having sample space S, then



$$\mathbf{P(A \cup B) = P(A) + P(B)} \quad \text{(axiom } (iii))$$

For example, let S be the sample space associated with tossing a fair coin 3 times, then

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT},

which has 8 equally likely outcomes.

Let A = getting three heads = {HHH} and

B = getting exactly two heads = {HHT, HTH, THH}.

Here A ∪ B = {HHH, HHT, HTH, THH}, and we note that A and B have no outcome in common, so these are mutually exclusive events.

We note that $P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$, $P(B) = \frac{n(B)}{n(S)} = \frac{3}{8}$ and $P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{4}{8}$.

Obviously, $P(A) + P(B) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = P(A \cup B)$, which verifies the theorem.

Generalised form of the above result is:

If $A_1, A_2, \ldots, A_n$ are mutually exclusive events associated with a random experiment, then

$$\mathbf{P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).}$$

Note that $A_1 \cup A_2 \ldots \cup A_n$ means A_1 or A_2 ... or A_n *i.e.* atleast one of events $A_1, A_2, \ldots, A_n$ happens.

By using principle of induction, we can easily prove that if $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

Addition theorem for mutually exclusive and exhaustive events

Theorem 2. *If A and B are mutually exclusive and exhaustive events associated with a random experiment having sample space S, then P(A) + P(B) = 1.*

Proof. Since A and B are mutually exclusive, so

$$P(A \cup B) = P(A) + P(B) \quad \ldots(i)$$

Also A and B are exhaustive, so $A \cup B = S$

$\Rightarrow \quad P(A \cup B) = P(S) = 1$

$\Rightarrow \quad P(A) + P(B) = 1 \quad$ (using (*i*))

Generalised form of the above theorem is :

If $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive events associated with a random experiment having sample space S, then

$$P(A_1) + P(A_2) + \ldots + P(A_n) = 1.$$

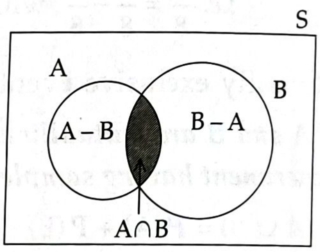
For example, let A_1 = {any spade card}, A_2 = {any heart card},
A_3 = {any diamond card}, A_4 = {any club card},

then A_1, A_2, A_3, A_4 are mutually exclusive and exhaustive events and

$$P(A_1) + P(A_2) + P(A_3) + P(A_4) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1.$$

Addition theorem on probabilities

Theorem 3. *If A and B are any two events associated with a random experiment having sample space S, then*
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.



Proof. From the adjoining Venn diagram, we have

$A \cup B = A \cup (B - A)$

$\Rightarrow \quad P(A \cup B) = P(A \cup (B - A))$

$\Rightarrow \quad P(A \cup B) = P(A) + P(B - A)$

(Using axiom (*iii*), as A and B – A are mutually exclusive)

$\Rightarrow \quad P(B - A) = P(A \cup B) - P(A) \quad \ldots(1)$

Also, $B = (A \cap B) \cup (B - A)$

$\Rightarrow \quad P(B) = P((A \cap B) \cup (B - A))$

$\Rightarrow \quad P(B) = P(A \cap B) + P(B - A)$

(Using axiom (*iii*), as $A \cap B$ and B – A are mutually exclusive)

$\Rightarrow \quad P(B - A) = P(B) - P(A \cap B) \quad \ldots(2)$

From (1) and (2), we have

$P(A \cup B) - P(A) = P(B) - P(A \cap B)$

$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

For example, consider the experiment of tossing a coin thrice, so that sample space

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT},

which has 8 equally likely outcomes.

Let A = getting two or more heads = {HHH, HHT, HTH, THH} and

B = getting an odd number of heads = {HHH, HTT, THT, TTH}

We note that A and B are not mutually exclusive as they have one outcome HHH in common.

Here $A \cap B = \{HHH\}$, $A \cup B = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{4}{8} = \frac{1}{2}, P(B) = \frac{4}{8} = \frac{1}{2},$$

$$P(A \cap B) = \frac{1}{8} \text{ and } P(A \cup B) = \frac{7}{8}.$$

We can verify that as predicted by the theorem,

$$P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{8} = \frac{7}{8} = P(A \cup B).$$

Theorem 4. ***If A and B are any two events associated with a random experiment, then***

(i) $P(A \cup B) = P(A - B) + P(B - A) + P(A \cap B)$

(ii) $P(A \cup B) = P(A) + P(B - A) = P(B) + P(A - B)$

(iii) $P(A) = P(A - B) + P(A \cap B)$ ***(iv)*** $P(B) = P(B - A) + P(A \cap B)$

(v) $P(A - B) = P(A \cap \bar{B})$ ***(vi)*** $P(B - A) = P(\bar{A} \cap B)$

(vii) $P(A) + P(B) = P(A - B) + P(B - A) + 2P(A \cap B)$.

Remarks

1. As $A \subset A \cup B$, $B \subset A \cup B$, so $P(A) \le P(A \cup B)$, $P(B) \le P(A \cup B)$.
2. As $A \cap B \subset A$, $A \cap B \subset B$, so $P(A \cap B) \le P(A)$, $P(A \cap B) \le P(B)$.
3. $P(A \cap \bar{B})$ or $P(A - B)$ is known as the probability of occurrence of A only.
4. $P(\bar{A} \cap B)$ or $P(B - A)$ is known as the probability of occurrence of B only.
5. $P((A \cap \bar{B}) \cup (\bar{A} \cap B))$ is known as the probability of occurrence of only (or exactly) one of the two events A and B.

   Since the events $A - B$ and $B - A$ *i.e.* the events $A \cap \bar{B}$ and $\bar{A} \cap B$ are mutually exclusive, so $P((A \cap \bar{B}) \cup (\bar{A} \cap B)) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$.

Thus, the probability of occurrence of only (or exactly) one of the two events A and B is given by $P(A \cap \bar{B}) + P(\bar{A} \cap B)$.

Theorem 5. ***If A, B and C are any three events associated with a random experiment, then***

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Proof. $P(A \cup B \cup C) = P((A \cup B) \cup C)$

$= P(A \cup B) + P(C) - P((A \cup B) \cap C)$

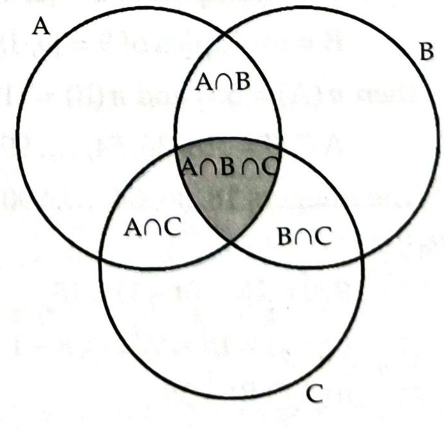
(using theorem 3)

$= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cup (B \cap C))$

$= P(A) + P(B) + P(C) - P(A \cap B) - [P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))]$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

$(\because (A \cap C) \cap (B \cap C) = A \cap B \cap C)$



ILLUSTRATIVE EXAMPLES

Example 1. *A card is drawn at random from a deck of 52 playing cards. Find the probability that it is*

(*i*) *a three of clubs or a six of diamonds* (*ii*) *a ten or a spade*

(*iii*) *neither a four nor a club* (*iv*) *a spade or a picture card.*

Solution. (*i*) P(a three of clubs or a six of diamonds) = $\frac{2}{52} = \frac{1}{26}$.

Alternatively, as getting a three of clubs or a six of diamonds are mutually exclusive events, we have

$$P(3C \cup 6D) = P(3C) + P(6D) = \frac{1}{52} + \frac{1}{52} = \frac{1}{26}.$$

(*ii*) There are four 10's and 13 spades but there is one double counting, that of 10S.

$\therefore$ total number of such cards = 4 + 13 – 1 = 16.

$\therefore$ $P(10 \cup S) = \frac{16}{52} = \frac{4}{13}$.

Alternatively, as events 10's and spade's are not mutually exclusive,

$$P(10 \cup S) = P(10) + P(S) - P(10 \cap S) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

(*iii*) "neither a four nor a club" means not a (four or a club)

Now $P(4 \cup C) = P(4) + P(C) - P(4 \cap C) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$.

$\therefore$ $P(\text{not } (4 \cup C)) = 1 - \frac{4}{13} = \frac{9}{13}$.

Alternatively,

P(neither a four nor a club) = $P(4' \cap C') = P[(4 \cup C)']$

$$= 1 - P(4 \cup C) = 1 - \frac{4}{13} = \frac{9}{13}.$$

(*iv*) There are 13 spades and 9 other picture cards (3 of hearts, diamonds and clubs each). So

P(a spade or a picture card) = $\frac{22}{52} = \frac{11}{26}$.

Example 2. *If an integer from 1 to 1000 is chosen at random, find the probability that the integer is a multiple of 2 or 9.*

Solution. Here S = {1, 2, 3, ..., 1000}, which has 1000 equally likely outcomes.

Let events A and B be defined as follows :

A = multiples of 2 = {2, 4, 6, ..., 1000} and

B = multiples of 9 = {9, 18, 27, ..., 999},

then $n(A) = 500$ and $n(B) = 111$.

$A \cap B = \{18, 36, 54, ..., 990\}$.

The integers 18, 36, 54, ..., 990 from an A.P. with $a = 18$, $d = 18$ and $l = 990$. If number of terms is n, then

$990 = 18 + (n - 1) \times 18$ $\quad |\, l = a + (n-1)\,d$

$\Rightarrow$ $(n - 1) \times 18 = 972 \Rightarrow n - 1 = 54 \Rightarrow n = 55$

$\Rightarrow$ $n(A \cap B) = 55$.

$$P(A) = \frac{500}{1000},\ P(B) = \frac{111}{1000} \text{ and } P(A \cap B) = \frac{55}{1000}.$$

$$\text{P(multiple of 2 or 9)} = P(A \cup B)$$
$$= P(A) + P(B) - P(A \cap B)$$
$$= \frac{500}{1000} + \frac{111}{1000} - \frac{55}{1000} = \frac{556}{1000} = 0.556$$

Example 3. *The probability of an event A occurring is 0.5 and of B is 0.3. If A and B are mutually exclusive events, then find the probability of neither A nor B occurring.*

Solution. Given events A and B are mutually exclusive,

$$P(A \cup B) = P(A) + P(B) = 0.5 + 0.3 = 0.8$$

$\therefore$ P(neither A nor B) $= P(A' \cap B') = P((A \cup B)')$

$$= 1 - P(A \cup B) = 1 - 0.8 = 0.2$$

Example 4. *Events E and F are such that P (not E or not F) = 0.25. State whether E and F are mutually exclusive.*

Solution. Given P(not E or not F) = 0.25 $\Rightarrow P(E' \cup F') = 0.25$

$\Rightarrow \quad P((E \cap F)') = 0.25 \Rightarrow 1 - P(E \cap F) = 0.25$

$\Rightarrow \quad P(E \cap F) = 1 - 0.25 \Rightarrow P(E \cap F) = 0.75$

$\Rightarrow \quad E \cap F \neq \phi \Rightarrow$ E and F are not mutually exclusive.

Example 5. *The probability of an event A occurring is $\frac{2}{3}$ and the probability of event B not occurring is $\frac{5}{9}$. If the probability of getting success in atleast one of the two events is $\frac{4}{5}$, what is the probability of success in both the events?*

Solution. Given $P(A) = \frac{2}{3}$, $P(\bar{B}) = \frac{5}{9}$ and $P(A \cup B) = \frac{4}{5}$.

Now $P(B) = 1 - P(\bar{B}) = 1 - \frac{5}{9} = \frac{4}{9}$.

We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$= \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{30 + 20 - 36}{45} = \frac{14}{45}.$$

Example 6. *A and B are two events such that $P(A) = 0.54$, $P(B) = 0.69$ and $P(A \cap B) = 0.35$, find*

(i) $P(A \cup B)$ *(ii)* $P(A' \cap B')$ *(iii)* $P(A \cap B')$ *(iv)* $P(B \cap A')$.

Solution. Given P(A) = 0.54, P(B) = 0.69 and $P(A \cap B) = 0.35$

(i) We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \quad P(A \cup B) = 0.54 + 0.69 - 0.35 = 0.88$

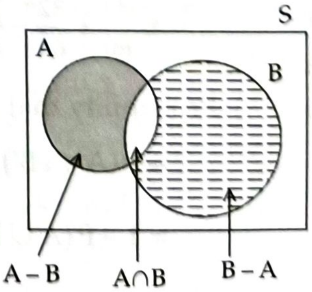
(ii) $P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B)$

$= 1 - 0.88 = 0.12$

(iii) $P(A \cap B') = P(A - B)$

$= P(A) - P(A \cap B)$ (from Venn diagram)

$= 0.54 - 0.35 = 0.19$



(iv) $P(B \cap A') = P(B - A)$

$= P(B) - P(A \cap B)$ (from Venn diagram)

$= 0.69 - 0.35 = 0.34$

Example 7. *Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05, and that Ashima will qualify is 0.10. The probability that both will qualify is 0.02. Find the probability that*

(i) *both Anil and Ashima will not qualify the exam,*

(ii) *atleast one of them will not qualify the exam, and*

(iii) *only one of them will qualify the exam.*

Solution. Let A and B denote the events that Anil and Ashima will respectively qualify the exam, then

$$P(A) = 0.05, P(B) = 0.10, P(A \cap B) = 0.02$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.05 + 0.10 - 0.02 = 0.13$$

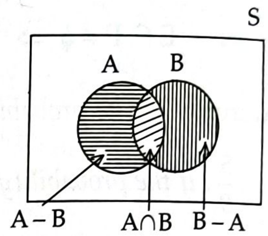
(i) The event that both Anil and Ashima will not qualify the exam can be expressed as none qualifies *i.e.* as $A' \cap B'$.

Now $P(A' \cap B') = P[(A \cup B)'] = 1 - P(A \cup B) = 1 - 0.13 = 0.87$

(ii) Probability of atleast one of them does not qualify $= P(A' \cup B') = P((A \cap B)')$

$$= 1 - P(A \cap B) = 1 - 0.02 = 0.98$$

(iii) Given event, that only one of them qualifies, is expressed as union of $A - B$ and $B - A$. But the events $A - B$ and $B - A$ are mutually exclusive, therefore,



$$P((A - B) \cup (B - A)) = P(A - B) + P(B - A).$$

$\therefore$ Required probability $= P(A - B) + P(B - A)$

$= P(A \cup B) - P(A \cap B)$ (from Venn diagram)

$= 0.13 - 0.02 = 0.11$

Example 8. *In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that*

(i) *the student opted for NCC or NSS.*

(ii) *the student opted neither NCC nor NSS.*

(iii) *the student has opted for NSS but not NCC.*

Solution. Let A = event of student opting for NCC and

B = event of student opting for NSS.

Then $P(A) = \dfrac{30}{60}, P(B) = \dfrac{32}{60}$ and $P(A \cap B) = \dfrac{24}{60}$.

(i) Probability that the student opted for NCC or NSS

$$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{30}{60} + \frac{32}{60} - \frac{24}{60} = \frac{30 + 32 - 24}{60} = \frac{38}{60} = \frac{19}{30}.$$

(ii) Probability that the student opted neither NCC nor NSS

$$= P(A' \cap B') = P((A \cup B)')$$

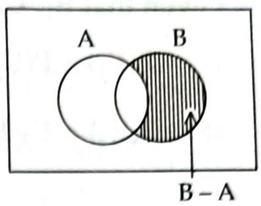
$$= 1 - P(A \cup B) = 1 - \frac{19}{30} = \frac{11}{30}.$$

(iii) Probability that the student has opted for NSS but not NCC

$= P(B - A)$

$= P(B) - P(A \cap B)$ (from Venn diagram)

$= \frac{32}{60} - \frac{24}{60} = \frac{8}{60} = \frac{2}{15}$.



Example 9. *Find the probability of atmost two tails or atleast two heads in a toss of three coins.*

Solution. Let S be the sample space for the experiment of a toss of three coins, then

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

Let A and B be the events defined as follows :

A = getting atmost two tails in a toss of three coins, and

B = getting atleast two heads in a toss of three coins.

Then A = {HHH, HHT, HTH, HTT, THH, THT, TTH} and

B = {HHH, HHT, HTH, THH}.

∴ $A \cap B$ = {HHH, HHT. HTH, THH}.

Here, $P(A) = \frac{7}{8}$, $P(B) = \frac{4}{8} = \frac{1}{2}$, $P(A \cap B) = \frac{4}{8} = \frac{1}{2}$.

∴ Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{7}{8} + \frac{1}{2} - \frac{1}{2} = \frac{7}{8}$.

Example 10. *Two dice are rolled once. Find the probability of getting an odd number on the first die or a total of 8.*

Solution. When two dice are rolled once, then the sample space has 36 equally likely outcomes.

Let A be the event 'odd number on the first die' and B be the event 'a total of 8'. Then

A = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)}

and B = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}.

Note that $A \cap B$ = {(3, 5), (5, 3)}.

Here $n(A) = 18$, $n(B) = 5$ and $n(A \cap B) = 2$.

∴ $P(A) = \frac{18}{36}$, $P(B) = \frac{5}{36}$ and $P(A \cap B) = \frac{2}{36}$.

∴ Required probability = P(an odd number on first die or a total of 8)

$= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{18}{36} + \frac{5}{36} - \frac{2}{36} = \frac{21}{36} = \frac{7}{12}$.

Example 11. *There are three mutually exclusive and exhaustive events E_1, E_2 and E_3. The odds are 8 : 3 against E_1 and 2 : 5 in favour of E_2. Find the odds against E_3.*

Solution. We know that if odds in favour are $a : b$, the probability is $\frac{a}{a+b}$.

Since odds in favour of E_2 are 2 : 5, $P(E_2) = \frac{2}{2+5} = \frac{2}{7}$.

Odds against E_1 are 8 : 3. Alternatively, odds in favour of E_1 are 3 : 8

∴ $P(E_1) = \frac{3}{3+8} = \frac{3}{11}$.

Given that E_1, E_2 and E_3 are mutually exclusive and exhaustive

$\therefore \quad P(E_1) + P(E_2) + P(E_3) = 1$

$$\Rightarrow \quad P(E_3) = 1 - P(E_2) - P(E_1) = 1 - \frac{2}{7} - \frac{3}{11} = \frac{77 - 22 - 21}{77} = \frac{34}{77}.$$

$$\therefore \quad \text{Odds against } E_3 \text{ are } (1 - P(E_3)) : P(E_3) = \left(1 - \frac{34}{77}\right) : \frac{34}{77} = 43 : 34.$$

Example 12. *In a race, the probabilities of A and B winning the race are $\frac{1}{3}$ and $\frac{1}{6}$ respectively. Find the probability of neither of them winning the race.*

Solution. Given $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{6}$.

As both A and B cannot win the same race *i.e.* only one of them can win the race, the events A and B are mutually exclusive.

Now P(neither of them winning race) $= P(A' \cap B') = P((A \cup B)')$

$= 1 - P(A \cup B) = 1 - (P(A) + P(B))$

($\because$ A and B are mutually exclusive events)

$$= 1 - \left(\frac{1}{3} + \frac{1}{6}\right) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Example 13. *In a given race, the odds in favour of four horses A, B, C and D are 1 : 3, 1:4, 1 : 5 and 1 :6 respectively. Assuming that dead heat is impossible, find the chance that one of them wins the race.*

Solution. Let E_1, E_2, E_3 and E_4 be the events of winning the horses A, B, C and D respectively.

Since the odd of winning in favour of the horses A, B, C and D are 1 : 3, 1 : 4, 1 : 5 and 1 : 6 respectively,

$$\therefore \; P(E_1) = \frac{1}{4}, P(E_2) = \frac{1}{5}, P(E_3) = \frac{1}{6} \text{ and } P(E_4) = \frac{1}{7}.$$

As only one of the horses can win the race, the events E_1, E_2, E_3 and E_4 are mutually exclusive.

$\therefore$ The probability that one of them wins the race

$$= P(E_1 \cup E_2 \cup E_3 \cup E_4) = P(E_1) + P(E_2) + P(E_3) + P(E_4)$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$

$$= \frac{105 + 84 + 70 + 60}{420} = \frac{319}{420}.$$

Example 14. *An integer is chosen at random from the numbers 1 to 50. What is the probability that the integer chosen is a multiple of 2 or 3 or 10?*

Solution. Here, sample space S = {1, 2, 3, 4, …, 50}. It consists of 50 equally likely outcomes.

Let A, B and C be the events of getting multiples of 2, 3 and 10 respectively. Then

A = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50},

B = {3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48},

C = {10, 20, 30, 40, 50},

$A \cap B$ = {6, 12, 18, 24, 30, 36, 42, 48}, $B \cap C$ = {30},

$A \cap C$ = {10, 20, 30, 40, 50} and $A \cap B \cap C$ = {30}.

Here $n(A) = 25$, $n(B) = 16$, $n(C) = 5$, $n(A \cap B) = 8$, $n(B \cap C) = 1$, $n(A \cap C) = 5$ and $n(A \cap B \cap C) = 1$

$$\therefore \quad P(A) = \frac{25}{50}, P(B) = \frac{16}{50}, P(C) = \frac{5}{50}, P(A \cap B) = \frac{8}{50}, P(B \cap C) = \frac{1}{50}, P(A \cap C) = \frac{5}{50}$$

and $P(A \cap B \cap C) = \frac{1}{50}$.

Required probability $= P(A \cup B \cup C)$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

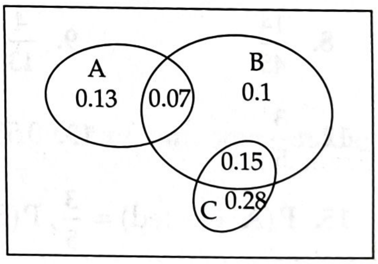
$$= \frac{25}{50} + \frac{16}{50} + \frac{5}{50} - \left(\frac{8}{50} + \frac{1}{50} + \frac{5}{50}\right) + \frac{1}{50}$$

$$= \frac{25 + 16 + 5 - (8 + 1 + 5) + 1}{50} = \frac{33}{50}.$$

EXERCISE 12.4

1. If $P(A \cup B) = P(A) + P(B)$, then what can be said about the events A and B?
2. Check whether the following probabilities $P(A)$ and $P(B)$ are consistently defined :
   (i) $P(A) = 0.5$, $P(B) = 0.7$, $P(A \cap B) = 0.6$
   (ii) $P(A) = 0.5$, $P(B) = 0.4$, $P(A \cup B) = 0.8$
3. If A and B are mutually exclusive events, $P(A) = 0.35$ and $P(B) = 0.45$, then find
   (i) $P(A')$ (ii) $P(B')$ (iii) $P(A \cup B)$ (iv) $P(A \cap B)$
   (v) $P(A \cap B')$ (vi) $P(A' \cap B')$

   Hint. (v) As A and B are mutually exclusive, $A - B = A$.
   $\therefore \; P(A \cap B') = P(A - B) = P(A) = 0.35$
4. If E and F are events such that $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and $P(E \text{ and } F) = \frac{1}{8}$, find
   (i) $P(E \text{ or } F)$ (ii) $P(\text{not } E \text{ and not } F)$.
5. If A, B and C are mutually exclusive and exhaustive events and it is known that $P(A \cup B) = 0.63$, calculate $P(C)$.
6. A, B, C are three mutually exclusive and exhaustive events associated with a random experiment. Find $P(A)$, it being given that $P(B) = \frac{3}{2} P(A)$ and $P(C) = \frac{1}{2} P(B)$.
7. The adjoining Venn diagram shows three events A, B and C, and also the probabilities of the various regions (for instance $P(A \cap B) = 0.07$). Determine



   (i) $P(A)$ (ii) $P(A \cup B)$
   (iii) $P(B \cap C)$ (iv) $P(B \cap \overline{C})$
   (v) $P(A \cap \overline{B})$
   (vi) Probability of exactly one of three events occurs.
8. The probability that a contractor will get a plumbing contract is $\frac{2}{3}$ and an electric contract is $\frac{4}{9}$. If the probability of getting atleast one contract is $\frac{4}{5}$, find the probability that he will get both contracts.
9. A card is drawn from a well shuffled pack of playing cards. What is the probability that it is either a spade or an ace or both?

10. A card is drawn at random from a pack of 52 playing cards. What is the probability that the card drawn is neither a spade nor a queen?

11. If two dice are thrown simultaneously, find the probability of getting
   (*i*) a sum of 7 or 11 (*ii*) a doublet or a total of 6.

12. In class XI of a school, 40% students are punctual and 30% students are regular. 10% of the students are both punctual and regular. If a student is selected at random from the class, find the probability that he will be punctual or regular or both.

13. In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

14. The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75, then what is the probability of passing the Hindi examination?

15. For a post three persons A, B and C appear in the interview. The probability of A being selected is twice that of B and the probability of B being selected is thrice that of C. If the post is filled, what are the probabilities of A, B and C being selected?

16. The probabilities that a student will get A, B, C or D grade are 0.4, 0.35, 0.15 and 0.1 respectively. Find the probability that she will get
   (*i*) B or C grade (*ii*) atmost C grade.

   Hint. (*i*) Events of getting A, B, C or D grades are mutually exclusive.
   (*ii*) atmost C grade means C or D grade.

17. Two cards are drawn at random from a pack of 52 cards. What is probability that the cards are either both aces or both black cards?

Answers

1. They are mutually exclusive (because $P(A \cap B) = 0$)
2. (*i*) No (because) $P(A \cap B)$ cannot be greater than $P(A)$) (*ii*) Yes
3. (*i*) 0.65 (*ii*) 0.55 (*iii*) 0.8 (*iv*) 0 (*v*) 0.35 (*vi*) 0.2
4. (*i*) $\frac{5}{8}$ (*ii*) $\frac{3}{8}$ 5. 0.37 6. $\frac{4}{13}$
7. (*i*) 0.2 (*ii*) 0.45 (*iii*) 0.15 (*iv*) 0.17 (*v*) 0.13 (*vi*) 0.51
8. $\frac{14}{45}$ 9. $\frac{4}{13}$ 10. $\frac{9}{13}$ 11. (*i*) $\frac{2}{9}$ (*ii*) $\frac{5}{18}$
12. $\frac{3}{5}$ 13. 0.55 14. 0.65
15. $P(A \text{ selected}) = \frac{3}{5}$, $P(B \text{ selected}) = \frac{3}{10}$, $P(C \text{ selected}) = \frac{1}{10}$
16. (*i*) 0.5 (*ii*) 0.25 17. $\frac{55}{221}$

CONDITIONAL PROBABILITY

Suppose you draw a card from a pack of 52 cards. The probability of its being a card of diamonds is $\frac{13}{52} = \frac{1}{4}$. Now suppose that you are told that the card drawn is a red card. Would the probability of it being a card of diamonds be $\frac{1}{4}$ as before or will it be different given this additional information? We know that there are only 26 red cards. So given that the card drawn is red, the probability of it being a diamond card is $\frac{13}{26} = \frac{1}{2}$. This changed probability is called **conditional probability**.

As another example, consider the event of tossing two coins once.

The sample space is S = {HH, HT, TH, TT}

Let A = event that two tails or two heads turn up,

and B = event that atleast one head turns up.

Then if we know that atleast one head turns up *i.e.* B has happened, what is the probability of A happening? Since we know that B has occurred, the set of possible outcomes is

$$B = \{HH, HT, TH\}.$$

Since these outcomes were equally likely in the original sample space S, we assign equal probabilities to each. Thus the original sample space S has been *reduced* to B. In the set B, the only outcome favourable to A is HH. Hence, given that B has occurred, the probability of A happening is $\frac{1}{3}$. We write this as P(A | B).

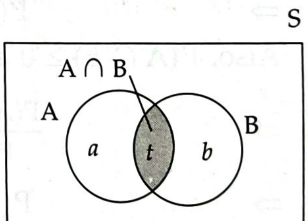
Let A and B be two events associated with a random experiment, then the probability of occurrence of the event A under the condition that B has occurred and P(B) ≠ 0 is called the conditional ***probability*** *and is denoted by P(A/B).*

Thus, P(A | B) = probability of occurrence of event A given that event B has already occurred, P(B) ≠ 0.

Similarly, P(B | A) = probability of occurrence of event B given that the event A has already occurred, P(A) ≠ 0.

In fact, the conditional probabilities P(A | B) and P(B | A) depend upon the nature of events A and B and also on the nature of the sample space S of the random experiment.

Let S be the sample space consisting of n equally likely outcomes of a random experiment. Let A and B be two events of the sample space S and a, b and t be the number of outcomes in A, B and A ∩ B respectively, then



$$P(A) = \frac{a}{n},\ P(B) = \frac{b}{n} \text{ and } P(A \cap B) = \frac{t}{n}.$$

Note that the outcomes of B which are favourable to A are the common outcomes of A and B *i.e.* the outcomes of A ∩ B.

$$\therefore\quad P(A \mid B) = \frac{\text{number of outcomes favourable to } A \cap B}{\text{number of outcomes favourable to } B} = \frac{t}{b}.$$

Dividing both numerator and denominator by O(S), the total number of outcomes in the sample space, we get

$$P(A \mid B) = \frac{\frac{t}{n}}{\frac{b}{n}} = \frac{P(A \cap B)}{P(B)}.$$

So $\mathbf{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$, P(B) ≠ 0.

Similarly, $\mathbf{P(B \mid A) = \frac{P(A \cap B)}{P(A)}}$**, P(A) ≠ 0.**

Remarks

1. In defining $P(A|B)$, we assume that $P(B) \neq 0$. If $P(B) = 0$, then B is an impossible event, so the assumption that B has occurred would be meaningless.
2. If the sample space S of a random experiment consists of equally likely outcomes, then in calculating $P(A|B)$, we basically calculate $P(A)$ with respect to the reduced sample space B, rather than with respect to the original sample space S.
3. $P(A|S) = \dfrac{P(A \cap S)}{P(S)} = \dfrac{P(A)}{1} = P(A)$
4. $P(A|A) = \dfrac{P(A \cap A)}{P(A)} = \dfrac{P(A)}{P(A)} = 1$
5. $P(S|A) = \dfrac{P(S \cap A)}{P(A)} = \dfrac{P(A)}{P(A)} = 1$

If the sample space S of a random experiment does not consist of equally likely outcomes, then we define P(A|B) as

$$P(A|B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0.$$

Similarly,

$$P(B|A) = \frac{P(A \cap B)}{P(A)},\ P(A) \neq 0.$$

Some properties of conditional probability

Property 1. *The conditional probability of an event A given that B has occurred lies between 0 and 1.*

Proof. We know that $A \cap B \subset B$

$$\therefore \quad P(A \cap B) \leq P(B)$$

$$\Rightarrow \quad \frac{P(A \cap B)}{P(B)} \leq 1 \quad (\text{if } P(B) \neq 0)$$

$$\Rightarrow \quad P(A|B) \leq 1.$$

Also, $P(A \cap B) \geq 0$ and $P(B) > 0$

$$\Rightarrow \quad \frac{P(A \cap B)}{P(B)} \geq 0$$

$$\Rightarrow \quad P(A|B) \geq 0$$

$$\therefore \quad 0 \leq P(A|B) \leq 1.$$

Property 2. *If A and B are any two events associated with a random experiment having sample space S, and F is an event such that $P(F) \neq 0$, then*

$$P(A \cup B|F) = P(A|F) + P(B|F) - P(A \cap B|F).$$

In particular, if A and B are disjoint events, then

$$P(A \cup B|F) = P(A|F) + P(B|F).$$

Proof. $P(A \cup B|F) = \dfrac{P((A \cup B) \cap F)}{P(F)}$ ...(*i*)

Now, by distributive law of union of sets over intersection,

$$P((A \cup B) \cap F) = P((A \cap F) \cup (B \cap F))$$
$$= P(A \cap F) + P(B \cap F) - P(A \cap B \cap F)$$

Putting this value of $P(A \cup B) \cap F)$ in (*i*), we get

$$P(A \cup B|F) = \frac{P(A \cap F)}{P(F)} + \frac{P(B \cap F)}{P(F)} - \frac{P((A \cap B) \cap F)}{P(F)}$$
$$= P(A|F) + P(B|F) - P(A \cap B|F).$$

When A and B are disjoint events, then $A \cap B = \phi$,

so $\quad P(A \cap B | F) = 0$

$\Rightarrow \quad P(A \cup B | F) = P(A|F) + P(B|F).$

Property 3. *$P(A'|B) = 1 - P(A|B)$.*

Proof. $P(S|B) = \dfrac{P(S \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

$\Rightarrow \quad P(A \cup A'|B) = 1 \qquad (\because S = A \cup A')$

$\Rightarrow \quad P(A|B) + P(A'|B) = 1 \qquad (\because$ A and A′ are disjoint$)$

$\Rightarrow \quad P(A'|B) = 1 - P(A|B)$

ILLUSTRATIVE EXAMPLES

Example 1. *If $P(A) = 0.8$, $P(B) = 0.5$ and $P(B|A) = 0.4$, then find*

(i) $P(A \cap B)$ *(ii)* $P(A|B)$ *(iii)* $P(A \cup B)$

Solution. *(i)* We know that $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

$$\Rightarrow \quad 0.4 = \frac{P(A \cap B)}{0.8} \Rightarrow P(A \cap B) = 0.4 \times 0.8 = 0.32$$

(ii) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.32}{0.5} = \dfrac{32}{50} = 0.64$

(iii) Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.8 + 0.5 - 0.32 = 0.98$$

Example 2. *If $P(A) = \dfrac{6}{11}$, $P(B) = \dfrac{5}{11}$ and $P(A \cup B) = \dfrac{7}{11}$, then find*

(i) $P(A \cap B)$ *(ii)* $P(A|B)$ *(iii)* $P(B|A)$

Solution. *(i)* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow \quad \frac{7}{11} = \frac{6}{11} + \frac{5}{11} - P(A \cap B) \Rightarrow P(A \cap B) = \frac{4}{11}.$$

(ii) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{4}{11}}{\frac{5}{11}} = \dfrac{4}{5}.$

(iii) $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{\frac{4}{11}}{\frac{6}{11}} = \dfrac{4}{6} = \dfrac{2}{3}.$

Example 3. *Evaluate $P(A \cup B)$ if $2P(A) = P(B) = \dfrac{5}{13}$ and $P(A|B) = \dfrac{2}{5}$.*

Solution. Given $P(B) = \dfrac{5}{13}$, $P(A|B) = \dfrac{2}{5}$ and

$$2P(A) = P(B) \Rightarrow P(A) = \frac{1}{2}P(B) = \frac{1}{2}.\frac{5}{13} = \frac{5}{26}.$$

We know that $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$\Rightarrow \quad P(A \cap B) = P(B).P(A|B) = \frac{5}{13}.\frac{2}{5} = \frac{2}{13}.$$

Now $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{5 + 10 - 4}{26} = \frac{11}{26}.$$

Example 4. *If A and B are events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$, then find :*

(i) $P(A|B)$ *(ii)* $P(A'|B)$ *(iii)* $P(A'|B')$

Solution. *(i)* $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{4}}{\frac{1}{3}} = \dfrac{3}{4}.$

(ii) We know that $P(A'|B) = 1 - P(A|B)$ (Property 3)

$\Rightarrow \quad P(A'|B) = 1 - \frac{3}{4} = \frac{1}{4}.$

(iii) To find $P(A'|B')$, we need $P(A' \cap B')$.

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}.$

$P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - \frac{7}{12} = \frac{5}{12}.$

$P(B') = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}.$

$$P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{\frac{5}{12}}{\frac{2}{3}} = \frac{5}{12} \times \frac{3}{2} = \frac{5}{8}.$$

Example 5. *10% of the bulbs produced in a factory are of red colour and 2% are red and defective. If one bulb is picked up at random, determine the probability of its being defective if it is red.*

Solution. Let A be the event of that bulb is of red colour and B be the event of bulb being defective, then we want to find $P(B|A)$.

Given $\quad P(A) = 10\% = \frac{10}{100} = \frac{1}{10}$ and

$P(A \cap B)$ = Probability of bulb of red colour and defective

$= 2\% = \frac{2}{100} = \frac{1}{50}.$

$$\therefore \quad P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{50}}{\frac{1}{10}} = \frac{1}{5}.$$

Example 6. *Ten cards numbered 1 to 10 are placed in a box, and one card is drawn randomly. If it is known that the number on the card drawn is more than 3, what is the probability that it is an even number?*

Solution. Let A be the event 'the number on the card drawn is even', and B be the event that 'the number on the card drawn is more than 3', we want to find $P(A|B)$.

Here, sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, which has 10 equally likely outcomes.

$$A = \{2, 4, 6, 8, 10\} \text{ and } B = \{4, 5, 6, 7, 8, 9, 10\}$$

Also $\quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{5}{10} = \dfrac{1}{2}$, $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{7}{10}.$

Now $\quad A \cap B = \{4, 6, 8, 10\},$

so $\quad P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{4}{10}.$

$\therefore \quad P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{4}{10}}{\frac{7}{10}} = \dfrac{4}{7}.$

Alternatively

Let A be the event 'the number on the card drawn is even' and B be the event that 'the number on the card drawn is more than 3'. We want to find P(A|B).

Then A = {2, 4, 6, 8, 10} and

B = {4, 5, 6, 7, 8, 9, 10}.

Since the original sample space has equally likely outcomes and B has 7 outcomes, out of which 4 outcomes (*viz.* 4, 6, 8, 10) are favourable to A, therefore

P(A|B) = probability of getting an even number on the card when the card drawn has a number greater than 3

$= \dfrac{4}{7}.$

Example 7. *A coin is tossed three times. Find P(E|F) if*

(*i*) *E : heads on third toss, F : heads on first two tosses*

(*ii*) *E : atleast two heads, F : atmost two heads*

(*iii*) *E : atmost two tails, F : atleast one tail.*

Solution. When a coin is tossed 3 times, then the sample space

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

It consists of 8 equally likely outcomes.

(*i*) E = {HHH, HTH, THH, TTH} and

F = {HHH, HHT}.

We note that out of 2 outcomes of F, the only outcome HHH of F is favourable to E,

$\therefore \quad P(E|F) = \dfrac{1}{2}.$

(*ii*) E = {HHH, HHT, HTH, THH} and

F = {HHT, HTH, HTT, THH, THT, TTH, TTT}.

We note that out of 7 outcomes of F, three outcomes HHT, HTH, THH of F are favourable to E.

$\therefore \quad P(E|F) = \dfrac{3}{7}.$

(*iii*) E = {HHH, HHT, HTH, HTT, THH, THT, TTH} and

F = {HHT, HTH, HTT, THH, THT, TTH, TTT}.

We note that out of 7 outcomes of F, six outcomes HHT, HTH, HTT, THH, THT, TTH of F are favourable to E.

$\therefore \quad P(E|F) = \dfrac{6}{7}.$

Example 8. *A black and a red die are rolled.*

Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

Solution. Let S = {(*x*, *y*) : *x* is a number on the black die and *y* is a number on the red die}, then the sample space consists of 36 equally likely outcomes.

Let A be the event 'sum is greater than 9' *i.e.* the sum is 10, 11 or 12, and B be the event' black die resulted in a 5', then

$A = \{(4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}$ and

$B = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$.

We note that out of 6 outcomes of B, 2 outcomes (5, 5) and (5, 6) of B are favourable to A.

$\therefore \quad P(A \mid B) = \frac{2}{6} = \frac{1}{3}.$

Example 9. *Given that the numbers appearing on throwing two dice are different. Find the probability of the event 'the sum of the numbers on the dice is 4'.*

Solution. When two dice are thrown, the sample space consists of 36 equally likely outcomes. Let A be the event 'the numbers appearing on the two dice are different' and B be the event 'the sum of numbers on the dice is 4', then

$A = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 3), (2, 4), (2, 5), (2, 6),$
$(3, 1), (3, 2), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6),$
$(5, 1), (5, 2), (5, 3), (5, 4), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)\}$

and $B = \{(1, 3), (3, 1), (2, 2)\}$.

We note that out of 30 outcomes of A, two outcomes (1, 3) and (3, 1) are favourable to B.

$\therefore \quad P(B \mid A) = \frac{2}{30} = \frac{1}{15}.$

Example 10. *In a group of 100 sports car buyers, 40 bought alarm system, 30 purchased bucket seats and 20 purchased an alarm system and bucket seats. If a car buyer chosen at random, bought an alarm system, what is the probability that he also bought bucket seats?*

Solution. Let A be the event 'car buyer bought alarm system' and B be the event 'car buyer bought bucket seat'.

Given total number of sports car buyers = 100

$n(A) = 40, n(B) = 30, n(A \cap B) = 20$

$\therefore \quad P(A) = \frac{40}{100} = 0.4$ and $P(A \cap B) = \frac{20}{100} = 0.2$

$\therefore \quad P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.2}{0.4} = \frac{1}{2}.$

Example 11. *From the given data, find out the probability that a randomly selected person is male, given that he owns a pet.*

	Have pets	*Do not have pets*	*Total*
Male	0.41	0.08	0.49
Female	0.45	0.06	0.51
Total	0.86	0.14	1

Solution. Let A be the event 'Person is male' and B be the event 'Person owns a pet'.

$n(A) = 0.49, n(B) = 0.86, n(A \cap B) = 0.41$

So, $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.41}{0.86} = 0.477$

Example 12. *In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in class XII given that the chosen student is a girl?*

Solution. As the school has 1000 students and one student is chosen randomly, so the sample space S has 1000 equally likely outcomes.

Let A be the event 'a girl is chosen' and B be the event 'she studies in class XII'.

As the number of girls in the school is 430, 10% of the girls study in class XII, so the number of girls who study in class XII = 10% of 430 = $\frac{10}{100} \times 430 = 43$.

Thus, $n(\mathrm{A}) = 430$ and $n(\mathrm{A} \cap \mathrm{B}) = 43$

$\Rightarrow \; P(A) = \dfrac{430}{1000} = \dfrac{43}{100}$ and $P(A \cap B) = \dfrac{43}{1000}$.

$$\therefore \; P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{43}{1000}}{\frac{43}{100}} = \frac{1}{10} = 0.1$$

Example 13. *An instructor has a question bank consisting of 300 easy True/False questions, 200 difficult True/False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, then what is the probability that it will be an easy question given that it is a multiple choice question?*

Solution. Total number of questions in the question bank = 300 + 200 + 500 + 400 = 1400.

As a question is selected at random, so the sample space has 1400 equally likely outcomes.

Let A = a multiple choice question is selected and

B = an easy question is selected, then

$A \cap B$ = an easy multiple choice question is selected.

$n(A) = 500 + 400 = 900$ and $n(A \cap B) = 500$.

$\therefore \; P(A) = \dfrac{900}{1400} = \dfrac{9}{14}$ and $P(A \cap B) = \dfrac{500}{1400} = \dfrac{5}{14}$.

$$\therefore \; P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{5}{14}}{\frac{9}{14}} = \frac{5}{9}.$$

Example 14. *An electronic assembly consists of two subsystems A and B. Past testing procedures data show that probabilities of failure are*

P(A fails) = 0.2, P(B fails alone) = 0.15, P(A and B fail) = 0.15.

Evaluate (i) P(A fails | B has failed) (ii) P(A fails alone).

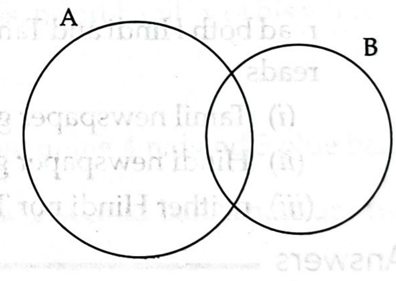
Solution. Given P(A fails) = 0.2 *i.e.* $P(\overline{A}) = 0.2$

$\Rightarrow \quad P(A) = 1 - P(\overline{A}) = 1 - 0.2 = 0.8$

P(B fails alone) = 0.15

$\Rightarrow \quad P(A \cap \overline{B}) = 0.15 \Rightarrow P(A - B) = 0.15$

Also P(A and B fail) = 0.15 $\Rightarrow P(\overline{A} \cap \overline{B}) = 0.15$



$\Rightarrow \quad P(\overline{A \cup B}) = 0.15 \Rightarrow 1 - P(A \cup B) = 0.15$

$\Rightarrow \quad P(A \cup B) = 1 - 0.15 = 0.85$

From the Venn diagram, $A \cup B = B \cup (A - B)$

$\Rightarrow \quad P(A \cup B) = P(B \cup (A - B))$

$\Rightarrow \quad P(A \cup B) = P(B) + P(A - B)$ $\qquad$ ($\because$ B and A – B are mutually disjoint)

$\Rightarrow \quad 0.85 = P(B) + 0.15 \Rightarrow P(B) = 0.7$

$\Rightarrow \quad P(\overline{B}) = 1 - P(B) = 1 - 0.7 = 0.3$

(*i*) P(A fails | B has failed) $= P(\overline{A} \mid \overline{B}) = \dfrac{P(\overline{A} \cap \overline{B})}{P(\overline{B})}$

$= \dfrac{0.15}{0.3} = \dfrac{15}{30} = \dfrac{1}{2} = 0.5.$

(*ii*) From Venn diagram, $A \cup B = A \cup (B - A)$

$\Rightarrow \quad P(A \cup B) = P(A \cup (B - A))$

$\Rightarrow \quad P(A \cup B) = P(A) + P(B - A)$ $\qquad$ ($\because$ A and B – A are mutually disjoint)

$$\Rightarrow \quad 0.85 = 0.8 + P(B \cap \overline{A})$$

$$\Rightarrow \quad P(B \cap \overline{A}) = 0.05$$

$$\Rightarrow \quad P(\text{A fails alone}) = 0.05$$

EXERCISE 12.5

1. (*i*) If $P(A) = \frac{1}{2}$, $P(B) = 0$, then what can you say about $P(A|B)$?

   (*ii*) If A and B are events such that $P(A|B) = P(B|A)$, then show that $P(A) = P(B)$.

2. If $P(\text{not } A) = 0.7$, $P(B) = 0.7$ and $P(B|A) = 0.5$, then find $P(A|B)$ and $P(A \cup B)$.

3. Compute $P(A|B)$ when $P(A) = \frac{1}{5}$, $P(B) = \frac{2}{5}$, $P(A \cup B) = \frac{3}{5}$.

4. A die is thrown twice and the sum of numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared atleast once?

5. A pair of dice is thrown and the sum of the numbers is observed to be even. What is the probability that both dice have come up with even numbers?

6. A family has two children. What is the probability that both the children are boys given that atleast one of them is a boy?

7. One card is drawn from a well-shuffled pack of 52 cards. If E is the event 'the card drawn is a king or a queen' and F is the event 'the card drawn is a queen or an ace', then find the probability of conditional event E/F.

8. I roll two dice and get a sum more than 9. What is the probability that the number on the first die is even?

9. In a certain school, 20% of the students failed in English, 15% of the students failed in Mathematics and 10% of the students failed in both English and Mathematics. A student is selected at random. If he passed in English, what is the probability that he also passed in Mathematics?

10. 60% students read Hindi newspaper, 40% students read Tamil newspaper and 20% students read both Hindi and Tamil newspaper. Find the probability that a student selected at random reads

    (*i*) Tamil newspaper given that he has already read Hindi newspaper.

    (*ii*) Hindi newspaper given that he has already read Tamil newspaper.

    (*iii*) neither Hindi nor Tamil newspaper.

Answers

1. (*i*) not defined

2. $P(A|B) = \frac{3}{14}$, $P(A \cup B) = 0.85$

3. 0

4. $\frac{2}{5}$

5. $\frac{1}{2}$

6. $\frac{1}{3}$

7. $\frac{1}{2}$

8. $\frac{2}{3}$

9. $\frac{15}{16}$

10. (*i*) $\frac{1}{3}$ (*ii*) $\frac{1}{2}$ (*iii*) $\frac{1}{5}$